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A Curious Property of Three-by-Three Matrices and an Application to Orders

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Let a three-by-three real matrix be given. Form the six possible row differences and add their maximum components. Next do the same for the columns. The result is the same.[1] We illustrate with the matrix

$$\begin{pmatrix} 6 & 1 & -3 \\ -1 & -5 & 0 \\ 7 & -2 & 4 \end{pmatrix}$$

bearing in mind that the maximum component of the ith minus the jth row coincides with the negative of the minimum component of the jth minus the ith.

Row differences:

				max	$-$ min
1st $-$ 2nd	7,	6,	-3	7	3
2nd $-$ 3rd	-8,	-3,	-4	-3	8
3rd $-$ 1st	1,	-3,	7	7	3
				11	14

Column differences:

				max	$-$ min
1st $-$ 2nd	5,	4,	9	9	-4
2nd $-$ 3rd	4,	-5,	-6	4	6
3rd $-$ 1st	-9,	1,	-3	1	9
				14	11

In both cases the sum is 25.

For a given three-by-three matrix, we may speak of the contribution from each of its elements to the two sums we are comparing. In the example above, since $-5+2$ is the maximum component of the 2nd minus the 3rd row, and $-2-1$ the minimum component of the 3rd minus the 1st

[1] This curious fact was conjectured by the second author in connection with the question on orders described later in this note. He proved it by a crude method that involved a very large number of case distinctions. On hearing this mentioned at a talk at Queen's University, the first author came up with the following proof.

row, there is a contribution of $+ (+2) - (-2) = 4$ from the matrix element -2 to the sum for rows. Our proof will consist in showing, for a given matrix, that the contributions from each of its elements to the two sums are the same.

Suppose the given matrix is

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

We consider a two-by-two block of adjacent elements, say

$$\begin{matrix} a & b \\ d & e, \end{matrix}$$

and form the expression $a + e - b - d$. Imagining the matrix elements repeating themselves as on a torus, we form all such expressions. Let us first suppose that none of them is zero. The crucial observation is that the contribution to each sum from a given matrix element can be determined explicitly from the signs of the four such expressions which involve that element. For instance, suppose that the signs of the expressions involving e are as in the array

$$\begin{matrix} + & - \\ - & -; \end{matrix}$$

that is, suppose that

$$a + e - b - d > 0,$$
$$b + f - c - e < 0,$$
$$d + h - e - g < 0,$$
$$e + i - f - h < 0.$$

Then we have the following comparisons for components of the row and column differences indicated.

$$\begin{array}{ll} \text{1st} - \text{2nd row,} & a - d > b - e < c - f; \\ \text{2nd} - \text{3rd row,} & d - g < e - h < f - i; \\ \text{1st} - \text{2nd column,} & a - b > d - e < g - h; \\ \text{2nd} - \text{3rd column,} & b - c < e - f < h - i. \end{array}$$

Thus $d - e$ and $b - e$ are minima, $e - f$ and $e - h$ are neither maxima nor minima, and we have from e a contribution of $+ e$ to each sum.

Since the calculation of the two sums is appropriately invariant under symmetries of the given matrix and change of sign, we have to consider only three other two-by-two arrays of signs:

$$\begin{matrix} - & - \\ - & - \end{matrix}, \qquad \begin{matrix} + & + \\ - & - \end{matrix}, \qquad \begin{matrix} + & - \\ - & + \end{matrix}.$$

An analysis similar to that above likewise yields for each case that the contributions e makes to the two sums are the same. Moreover, since permutations of rows and columns of the given matrix merely rearrange the six numbers in each sum, our treatment above for the center element applies equally well to any element.

If some of the expressions formed from the two-by-two blocks are zero, we begin by giving them signs. As long as we are careful to avoid rows or columns of the same sign (and there is no problem in doing this), we may then proceed as above. Such an assignment of signs corresponds to a choice among equals for the required maxima and minima.

The corresponding result for four-by-four or larger matrices is false. Almost any matrix will serve as an example, for instance,

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the discussion of the three-by-three case we assumed the entries of the matrix to be real numbers. However, the proof is valid when they belong to any linearly ordered abelian group (and the result is desirable in that generality).

Now for the application to orders. We follow the terminology of [2]. Let R be a valuation domain with quotient field K. Let L be a finite-dimensional algebra over K, associative and with unit element. We consider R-submodules of L, restricting attention to those which are finitely generated (therefore free) and span L as a vector space. If A is such a module, its *left order* P is the set of all x in L with $xA \subset A$, and its *right order* Q is analogously all y with $Ay \subset A$.

We ask: is there any relation between P and Q? Of course, if we assume enough, there is. If L is commutative, $P = Q$. If L is a $*$-algebra in the sense of [2] (and in particular a quaternion algebra, including two-by-two matrices as a special case), P and Q are conjugate.

A weaker notion than conjugacy is *unimodular equivalence*. Since P and Q, like A, are free modules of maximal rank, there exists a linear transformation mapping P onto Q. Its determinant is unique, up to a unit of R. If the determinant is itself a unit, we say that P and Q are unimodularly equivalent.

Now specialize L to be the algebra of all n by n matrices over K. We say that a module A is "tiled" if it has the form

$$\begin{pmatrix} Ra_{11} & \cdots & Ra_{1n} \\ \vdots & \cdots & \vdots \\ Ra_{n1} & \cdots & Ra_{nn} \end{pmatrix},$$

with the a_{ij}'s nonzero elements of K; this means that A consists of the matrices whose i, j-entry lies in Ra_{ij} for all i, j. (The term "tiled" was introduced in [3] as a translation of Faddeev's "kletochny," the AMS translation was "cellular.") If v is the valuation attached to R, the module A is determined by the matrix $\alpha_{ij} = v(a_{ij})$ of elements of the value group.

The left order P of A will also be tiled, say given by β_{ij}. It is immediate that

$$\beta_{ij} = \max_k(\alpha_{ik} - \alpha_{jk}),$$

i.e., one subtracts the jth row from the ith row and takes the maximum. The right order is given by elements γ_{ij} similarly determined from the columns. Unimodular equivalence means $\sum \beta_{ij} = \sum \gamma_{ij}$. It is now apparent that our problem is just the one proposed above. Thus we have: *for a tiled module inside the algebra of three-by-three matrices, the left and right orders are unimodularly equivalent.* Furthermore, the example above shows that this is false for larger matrices. For a number of related examples, see [1].

QUERY. Does unimodular equivalence hold for the left and right orders of three-by-three modules that are not tiled?

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